Szpiro's Conjecture

Samuele Anni, Sam Schiavone, Nicholas Triantafillou

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Conjecture

For each $\epsilon > 0$ there exists a constant C_ϵ such that if E is an elliptic curve over $\mathbb{Q}$ with minimal discriminant Δ and conductor N, then

$$|\Delta| \leq C_\epsilon N^{6+\epsilon}.$$

Definition

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- Szpiro's conjecture implies that σ is bounded.
- Szpiro's conjecture is equivalent to the statement:
  for all $M > 6$ there are only finitely many isomorphism classes of elliptic curves over $\mathbb{Q}$ such that $\sigma \geq M$.

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Conjecture

Let A, B, C be nonzero pairwise coprime integers with $A + B + C = 0$. For each $\epsilon > 0$, there exists a constant $\kappa(\epsilon) > 0$ such that

$$|ABC|^{1/3} < \kappa(\epsilon) N^{1+\epsilon}$$

where $N = \prod_{p|ABC} p$.

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Let K be a number field. There is a constant $c(K) > 0$ such that for all elliptic curves E/K and all non-torsion points $P \in E(K)$,

$$\widehat{h_E}(P) \geq c(K) \log(N_{K/\mathbb{Q}}(\Delta))$$

where $\widehat{h_E}$ is the canonical height on E.

Results In The Literature

- *L'ensemble exceptionnel dans la conjecture de Szpiro*, E. Fouvry, M. Nair, G. Tenenbaum
- *Détermination de courbes elliptiques pour la conjecture de Szpiro*, A. Nitaj

Results of Fouvry, et al.

- Show that Szpiro's conjecture holds for "almost all" elliptic curves.

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- Measure the density of the set of exceptions.

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For $a, b \in \mathbb{Z}$, let $E(a, b)$ be the elliptic curve given by $y^2 = x^3 + ax + b$.

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Let $S_0(A, B; M)$ be the number of pairs (a, b) such that

$$|a| \leq A, \ |b| \leq B, \quad \text{and} \quad \sigma_{E(a,b)} \geq M\,,$$

and such that $\nexists p$ prime with $p^4 \mid a$ and $p^6 \mid b$.

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Theorem

For any $M > 1$,

$$\lim_{A,B\to\infty} \frac{1}{AB} S_0(A, B; M) = 0\,.$$

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- Found the curve

$$E : y^2 + xy = x^3 + x^2 + 3494100111091075 72x - 7754287746183075058425565 92$$

with

$$\sigma_E = \frac{\log(2^{26} \cdot 3^{52} \cdot 5 \cdot 11^8 \cdot 13 \cdot 19^6 \cdot 31^4)}{\log(2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 31)} \approx 8.811944\,.$$

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3. By solving certain Diophantine equations, determine specific values of the parameters s, t that produce large Szpiro ratios.
4. Apply quadratic twists to try to further increase the Szpiro ratio.

Szpiro's Conjecture for Abelian Varieties

Generalized Szpiro (Hindry)

For $\varepsilon > 0$, there is a constant c_ε such that Falting's height and conductor of any abelian variety $A/\mathbb{Q}$ of dimension g satisfy

$$h_{\mathsf{Falt}}(A) \leq \left(\frac{g}{2} + \varepsilon\right) N_A + c_\varepsilon.$$

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Generalized Szpiro: Hyperelliptic Discriminant Version

There are constants c, κ such that if $C/\mathbb{Q}$ is a hyperelliptic curve of genus g, with Jacobian J, then $\Delta_C^{\mathrm{min}} \leq c_\varepsilon N_J^{\kappa+\varepsilon}$.

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Based on analogy with elliptic curves, and a "tentative suggestion" of Lockhart for a related conjecture, we tentatively suggest that $\kappa = 10 = 4g + 2$ might be the right value for genus 2.

Szpiro Ratio for Hyperelliptic Curves

Following Nitaj, we look for curves which force the constants in generalized Szpiro to be large.

Definition: Szpiro Ratio.

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The hyperelliptic discriminant version of generalized Szipiro would imply that for any fixed genus, σ is bounded.
For many 'random' curves we tried, σ is between 1 and 3.
To 'test' the conjectures, let's look for big σ.

Looking for Large Szpiro Ratios

For elliptic curves, wanted large isogeny classes

$\Rightarrow$ started with curves with large torsion.

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   - A not principally polarized.
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2. Even if $J(C_1) \sim J(C_2)$, different primes may divide $\Delta_{C_1}^{\min}$ and $\Delta_{C_2}^{\min}$.
   - C_1 has bad reduction at p but $J(C_1)$, C_2, and $J(C_2)$) have good reduction, and vice versa.

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But we want to experiment, and C with $\#J(C)(\mathbb{Q})_{\text{tors.}}$ large are interesting, so forge ahead.

First Experiments

We looked at several families of curves C_t with $\#J(C_t)(\mathbb{Q})$ large generically on the family and several sporadic examples.

- 13-torsion family (Flynn):

$$y^2 + (2tx - t)y = -t^2x^2(x-1)^3$$

- 15-torsion family (Leprevost):

$$y^2 = ((t+3)x^2 - (2t+3)x + t + 1)^2 - 4tx^3(x-1)^2$$

- 24-torsion family (Howe): See later slide
- Several others.

Howe 24-torsion family had much larger Szpiro ratios than the others.

Howe 24-Torsion Family

Constructed by "gluing elliptic curves along 2-torsion":

$$F : y^2 = g(x) = x^3 - 31x^2 + 256x$$
$$E_s : y^2 = f(x) = x^3 + \frac{-8(s^4 + 42s^2 - 147)}{(s^2 + 63)^2}x^2 + \frac{16(s^2 + 7)^3}{(s^2 + 63)^3}x$$

$J(C_s)$ is the image of $F \times E_s$ under a $(2, 2)$-isogeny ϕ, where $\ker(\phi)$ is the graph of an isomorphism of $F[2] \cong E_s[2]$ as Galois modules.

Equations for C_s can be given explicitly (and Howe does).

Howe 24-Torsion Family

For $s \in \mathbb{Q}$, define

$$\begin{aligned}
c_4 &= -31(s^4 + 42s^2 - (32200/93)s - 147) \\
c_2 &= 2^8(s^8 + 84s^6 - (3472/3)s^5 + 1470s^4 - 48608s^3 + 53508s^2 \\
&\qquad + 170128s + 21609) \\
c_0 &= 2^{20}(7/3)s(s^2 + 7)^3(s^2 + 63) \\
d &= s^4 + 42s^2 + (1736/3)s - 147
\end{aligned}$$

Then let $C : y^2 = (1/d)(x^6 + c_4x^4 + c_2x^2 + c_0)$.

Heuristic Explanation of Large Szpiro Ratios

1. Conductor (of the Jacobian) is nailed down. $J(C_s) \sim F \times E_s$, so $N_{J(C_s)} = N_F \cdot N_{E_s}$
   - Analogy to searching in isogeny families.
   - Conductor computation is provably correct and much easier.

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1. Conductor (of the Jacobian) is nailed down. $J(C_s) \sim F \times E_s$, so $N_{J(C_s)} = N_F \cdot N_{E_s}$
    - Analogy to searching in isogeny families.
    - Conductor computation is provably correct and much easier.
2. Large 'extra' prime factors often appear to high powers (≈ 20) in $\Delta(C_s)$.
    - J non-simple rules out an obstruction to such primes.
    - If J has good reduction at p and J_p is absolutely simple, then C has good reduction at p.

$s = 1/12^i$, $i = 1, \ldots, 8$

$\log\lvert\Delta\rvert / \log(N)$	$\approx$
$\frac{\log 2^{22}3^{6}7^{3}43^{1}211^{1}1009^{3}2042207^{22}}{\log 2^{2}3^{2}7^{2}43^{1}211^{1}1009^{1}}$	16.05
$\frac{\log 2^{28}3^{9}7^{3}23^{3}281^{1}4649^{1}6311^{3}61478548991^{22}}{\log 2^{2}3^{2}7^{2}23^{1}281^{1}4649^{1}6311^{1}}$	18.88
$\frac{\log 2^{34}3^{12}7^{3}317^{1}593429^{1}20901889^{3}1307680847585279^{22}}{\log 2^{2}3^{2}7^{2}317^{1}593429^{1}20901889^{1}}$	20.24
$\frac{\log 2^{40}3^{15}7^{3}19^{12}67^{1}51797^{3}58109^{3}404311147^{1}1430148767862371813^{22}}{\log 2^{2}3^{2}7^{2}67^{1}51797^{1}58109^{1}404311147^{1}}$	20.53
$\frac{\log 2^{46}3^{18}7^{3}3361^{3}6113^{12}15649^{1}128956129^{3}249267937^{1}9218940018932774191 9^{22}}{\log 2^{2}3^{2}7^{2}3361^{1}15649^{1}128956129^{1}249267937^{1}}$	20.28
$\frac{\log 2^{52}3^{21}7^{3}62412703137793^{3}561714328240129^{1}116860211328625544058026065 91^{22}}{\log 2^{2}3^{2}7^{2}62412703137793^{1}561714328240129^{1}}$	22.08
$\frac{\log 2^{58}3^{24}7^{3}41^{22}193^{1}293^{22}2567447^{1}163237223^{1}8987429251842049^{3}2017161633638299363031388188 3^{22}}{\log 2^{2}3^{2}7^{2}193^{1}2567447^{1}163237223^{1}8987429251842049^{1}}$	22.40
$\frac{\log 2^{64}3^{27}7^{3}89^{12}179^{12}211^{1}389^{1}653^{22}140533^{12}141908506565471^{1}1294189812265254913^{3}3437024817442027779629533^{12}}{\log 2^{2}3^{2}7^{2}211^{1}389^{1}141908506565471^{1}1294189812265254913^{1}}$	14.26

$s = 1/15^i$, $i = 1, \ldots, 8$

$\log \lvert\Delta\rvert / \log(N)$	$\approx$
$\frac{\log 2^{10} 3^{6} 5^{3} 7^{3} 197^{3} 443^{1} 5351^{22}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 197^{1} 443^{1}}$	11.49
$\frac{\log 2^{20} 3^{9} 5^{6} 7^{3} 11^{3} 4027^{3} 24917^{1} 23134463789^{12}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 11^{1} 4027^{1} 24917^{1}}$	12.30
$\frac{\log 2^{10} 3^{12} 5^{9} 7^{3} 11^{2} 23^{3} 37^{1} 317^{3} 1367^{3} 5009^{1} 107609^{22} 172884889^{12}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 11^{1} 23^{1} 37^{1} 317^{1} 1367^{1} 5009^{1}}$	14.81
$\frac{\log 2^{19} 3^{15} 5^{12} 7^{3} 23^{1} 29^{1} 127^{3} 227^{22} 1493^{3} 11827^{3} 392543^{12} 3782377^{1} 677190148049^{22}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 23^{1} 29^{1} 127^{1} 1493^{1} 11827^{1} 3782377^{1}}$	19.41
$\frac{\log 2^{10} 3^{18} 5^{15} 7^{3} 37^{3} 47^{12} 9013^{3} 170473^{1} 1513037^{3} 6659591^{1} 15927025913^{22} 63768729341^{12}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 37^{1} 9013^{1} 170473^{1} 1513037^{1} 6659591^{1}}$	16.10
$\frac{\log 2^{21} 3^{21} 5^{18} 7^{3} 151^{22} 66413419^{22} 1407420793^{12} 10957984217^{12} 31929762840271^{1} 113528045654297^{3}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 31929762840271^{1} 113528045654297^{1}}$	17.06
$\frac{\log 2^{10} 3^{24} 5^{21} 7^{3} 11^{3} 43^{3} 53^{3} 674059^{1} 900551^{3} 1131463^{3} 85264899827^{1} 2522629007334179^{12} 48497487117586519^{22}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 11^{1} 43^{1} 53^{1} 674059^{1} 900551^{1} 1131463^{1} 85264899827^{1}}$	17.50
$\frac{\log 2^{19} 3^{27} 5^{24} 7^{3} 11^{1} 193^{1} 241177^{22} 541129^{12} 8848351^{1} 64537223^{3} 344198033^{1} 3403520843^{12} 89054921239^{3} 892385549054469031^{22}}{\log 2^{2} 3^{2} 5^{1} 7^{2} 11^{1} 193^{1} 8848351^{1} 64537223^{1} 344198033^{1} 89054921239^{1}}$	19.50

Next Steps

1. Analyze the effect of taking quadratic twists/experiment with quadratic twists.
   - If J is semisimple, quadratic twisting shouldn't make Szpiro ratios above 5 larger (up to some possible funny business at 2.)
   - Quadratic twists by primes of good reduction move Szpiro ratio towards 2.5.
   - May be able to analyze additive or mixed reduction in particular families.

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   - If J is semisimple, quadratic twisting shouldn't make Szpiro ratios above 5 larger (up to some possible funny business at 2.)
   - Quadratic twists by primes of good reduction move Szpiro ratio towards 2.5.
   - May be able to analyze additive or mixed reduction in particular families.
2. Consider more families from 'gluing along torsion.'
3. Better understand when C has bad reduction but J has good reduction and construct large Szpiro examples.
4. Analytic argument à la Fouvry, Nair, Tenenbaum that almost all hyperelliptic curves (ordered by discriminant or coefficients) have Szpiro ratio close to one.

Thank you!